

# Multiple-Correction and Continued Fraction Approximation

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## Abstract

The main aim of this paper is to further develop a multiple-correction method formulated in a previous work [6]. As its applications, we find a kind of hybrid-type finite continued fraction approximations in two cases of Landau constants and Lebesgue constants. In addition, we refine the previous results of Lu [29] and Xu and You [44] for the Euler-Mascheroni constant.

## 1 Introduction

The constants of Landau and Lebesgue are defined for all integers  $n \geq 0$ , respectively, by

$$(1.1) \quad G(n) = \sum_{k=0}^n \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((n + \frac{1}{2})t)}{\sin(\frac{t}{2})} \right| dt.$$

The constants  $G(n)$  are important in complex analysis. In 1913, Landau [27] proved that if  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  is an analytic function in the unit disc satisfying  $|f(z)| < 1$  for  $|z| < 1$ , then  $|\sum_{k=0}^n a_k| \leq G(n)$ , and that this bound is optimal. Furthermore, Landau [27] showed that

$$(1.2) \quad G(n) \sim \frac{1}{\pi} \ln n, (n \rightarrow \infty).$$

Let  $\gamma$  denote the Euler-Mascheroni constant. In 1930, Watson [42] obtained a more precise asymptotic formula than (1.2)

$$(1.3) \quad G(n) \sim \frac{1}{\pi} \ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right), (n \rightarrow \infty),$$

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where

$$(1.4) \quad c_0 = \frac{1}{\pi}(\gamma + 4 \ln 2) = 1.0662758532089143543 \dots$$

In fact, the work of Watson opened up a novel insight into the asymptotic behavior of the Landau sequences  $(G(n))_{n \geq 0}$ . Inspired by (1.3), many authors investigated the upper and lower bounds of  $G(n)$ . We list some main results as follows:

$$(1.5) \quad \frac{1}{\pi} \ln(n+1) + 1 \leq G(n) < \frac{1}{\pi} \ln(n+1) + c_0 \quad (n \geq 0), \quad (\text{Brutman [5], 1982})$$

$$(1.6) \quad \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) + c_0 < G(n) \leq \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) + 1.0976 \quad (n \geq 0), \quad (\text{Falaleev [17], 1991})$$

$$(1.7) \quad \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) + c_0 < G(n) < \frac{1}{\pi} \ln \left( n + \frac{3}{4} + \frac{11}{192n} \right) + c_0 \quad (n \geq 1), \quad (\text{Mortici [34], 2011})$$

Recently, Chen [10] found the following better approximation for  $G(n)$ : as  $n \rightarrow \infty$ ,

$$(1.8) \quad G(n) = c_0 + \frac{1}{\pi} \ln \left( n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} - \frac{2009}{184320(n + \frac{3}{4})^3} + \frac{2599153}{371589(n + \frac{3}{4})^5} \right) + O \left( \frac{1}{(n + \frac{3}{4})^8} \right),$$

and the better upper bound:

$$(1.9) \quad G(n) < c_0 + \frac{1}{\pi} \ln \left( n + \frac{3}{4} + \frac{11}{192(n + \frac{3}{4})} \right), \quad (n \geq 0).$$

More recently, Cao, Xu and You [6] improved the rate of convergence to  $n^{-14}$ , and attained the following tight double-sides inequalities

$$(1.10) \quad \frac{C_1}{(n + \frac{3}{2})^6} < G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0 - \frac{\frac{11}{192\pi}}{(n + \frac{3}{4})^2 + \frac{1541}{7040}} < \frac{C_1}{(n + \frac{1}{2})^6}, \quad (n \geq 0),$$

where  $C_1 = \frac{89684299}{18166579200\pi}$ .

Another direction for developing the approximation to  $G(n)$  was initiated by Cvijović and Klinowski [12], who established the following estimates of  $G(n)$  in terms of the Psi(or Digamma) function  $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ :

$$(1.11) \quad \frac{1}{\pi} \psi \left( n + \frac{5}{4} \right) + c_0 < G(n) < \frac{1}{\pi} \psi \left( n + \frac{5}{4} \right) + 1.0725, \quad (n \geq 0),$$

$$(1.12) \quad \frac{1}{\pi} \psi \left( n + \frac{3}{2} \right) + 0.9883 < G(n) < \frac{1}{\pi} \psi \left( n + \frac{3}{2} \right) + c_0, \quad (n \geq 0).$$

Since then, many authors have made significant contributions to sharper the inequalities and the asymptotic expansions for  $G(n)$ , see e.g. Alzer [2], Chen [9], Cvijović and Srivastava [13], Granath [22], Mortici [34], Nemes [36, 37], Popa [38], Popa and Secelean [39], Zhao [46], Gavrea and M. Ivan [20], Chen and Choi [7, 10, 8], etc. To the best knowledge of the authors, the latest lower and upper bounds of  $G(n)$  along this research direction are due to Chen and Choi [8].

In 1906, Lebesgue [28] showed that if a function  $f$  is integrable on the interval  $[-\pi, \pi]$  and  $S_n(f; x)$  is the  $n$ -th partial sum of the Fourier series of  $f$ , then, we have

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \quad (k \geq 0) \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt,$$

$$S_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx)),$$

where the sum for  $n = 0$  is usually stipulated to be zero. If  $|f(x)| \leq 1$  for all  $x \in [-\pi, \pi]$ , then

$$(1.13) \quad \max_{x \in [-\pi, \pi]} |S_n(f; x)| \leq L_n \quad (n \geq 0).$$

It is noted that  $L_n$  is the smallest possible constant for which the inequality (1.13) holds for all integrable functions  $f$  on  $[-\pi, \pi]$ .

The Lebesgue constants play an important role in the theory of Fourier series. Therefore, they have attracted much attention of several well-known mathematicians such as Fejér [18], Gronwall [23], Hardy [24], Szegő [40], Watson [42], who established some remarkable properties of these numbers including monotonicity theorems, and various series and integral representations for  $L_n$ . Watson [42] showed

$$(1.14) \quad L_{n/2} = \frac{4}{\pi^2} \ln(n+1) + c_1 + O\left(\frac{1}{n^2}\right) \quad (n \rightarrow \infty),$$

where

$$(1.15) \quad c_1 = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{\ln k}{4k^2 - 1} + \frac{4}{\pi^2} (\gamma + 2 \ln 2) = 0.98943127383114695174 \dots$$

Since then, many authors have made important contributions to this research topic, see e.g. Galkin [19], Wong [43], Alzer [2], Zhao [46], Chen and Choi [11], etc. Let

$$(1.16) \quad u_n = L_{n/2} - \left( c_1 + \frac{4}{\pi^2} \psi \left( n + \frac{3}{2} + \frac{\frac{1}{8} - \frac{\pi^2}{72}}{n+1} \right) \right),$$

$$(1.17) \quad v_n = L_{n/2} - \left( c_1 + \frac{4}{\pi^2} \ln \left( n + 1 + \frac{a}{n+1} + \frac{c}{(n+1)^3} \right) \right),$$

where

$$a = \frac{1}{6} - \frac{\pi^2}{72} \quad \text{and} \quad c = -\frac{37}{360} + \frac{\pi^2}{135} + \frac{67\pi^4}{259200}.$$

Recently, Chen and Choi [7] obtained

$$(1.18) \quad \lim_{n \rightarrow \infty} n^4 u_n = \frac{-23625 + 1770\pi^2 + 67\pi^4}{64800\pi^2},$$

$$(1.19) \quad \lim_{n \rightarrow \infty} n^6 v_n = -\frac{-188637120 + 15135120\pi^2 + 308196\pi^4 + 7537\pi^6}{97977600\pi^2}.$$

**Notation.** Throughout the paper, the notation  $P_k(x)$  (or  $Q_k(x)$ ) as usual denotes a polynomial of degree  $k$  in terms of  $x$ . The notation  $\Psi(k; x)$  means a polynomial of degree  $k$  in terms of  $x$  with all of its non-zero coefficients being positive, which may be different at each occurrence. Notation  $\Phi(k; x)$  denotes a polynomial of degree  $k$  in terms of  $x$  with the leading coefficient being equal to one, which may be different at different subsections.

This paper is organized as follows. In Section 2, we prepare some preliminary lemmas. In Section 3, we explain how to find a finite continued fraction approximation by using the *multiple-correction method*. In Section 4 and Section 5, we discuss the constants of Landau and Lebesgue, respectively. In the last section, we consider to refine the works of Lu [29] and Xu and You [44] for the Euler-Mascheroni constant.

## 2 Some Preliminary Lemmas

The following lemma gives a method for measuring the rate of convergence, whose proof can be found in [31, 32].

**Lemma 1.** *If the sequence  $(x_n)_{n \in \mathbb{N}}$  is convergent to zero and there exists the limit*

$$(2.1) \quad \lim_{n \rightarrow +\infty} n^s (x_n - x_{n+1}) = l \in [-\infty, +\infty]$$

*with  $s > 1$ , then*

$$(2.2) \quad \lim_{n \rightarrow +\infty} n^{s-1} x_n = \frac{l}{s-1}.$$

In the study of Landau constants, we need to apply a so-called Brouncker's continued fraction formula.

**Lemma 2.** *For all integer  $n \geq 0$ , we have*

$$(2.3) \quad q(n) := \left( \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} \right)^2 = \frac{4}{1 + 4n + \frac{1^2}{2 + 8n + \frac{3^2}{2 + 8n + \frac{5^2}{2 + 8n + \ddots}}}}.$$

In 1654 when Brouncker and Wallis collaborated on the problem of squaring the circle, Lord William Brouncker found this remarkable fraction formula. Formula (2.3) was not published by Brouncker himself, and first appeared in [41]. For a general  $n$ , it actually follows from Entry

25 in Chapter 12 in Ramanujan's notebook [3], which gives a more general continued fraction formula for quotients of gamma functions.

Writing continued fractions in the way of (2.3) takes a lot of space and thus, we use the following shorthand notation

$$(2.4) \quad q(n) = \frac{4}{1+4n+} \frac{1^2}{2+8n+} \frac{3^2}{2+8n+} \frac{5^2}{2+8n+} \cdots = \frac{4}{1+4n+} \mathbf{K}_{k=0}^{\infty} \frac{(2k+1)^2}{2+8n},$$

and its  $k$ -th approximation  $q_k(n)$  is defined by

$$(2.5) \quad q_1(n) = \frac{4}{1+4n},$$

$$(2.6) \quad q_k(n) = \frac{4}{1+4n+} \frac{1^2}{2+8n+} \frac{3^2}{2+8n+} \frac{(2k-3)^2}{2+8n} = \frac{4}{1+4n+} \mathbf{K}_{j=0}^{k-1} \frac{(2j-1)^2}{2+8n}, \quad (k \geq 2).$$

**Lemma 3.** *Let  $c_1$  be defined by (1.15). Then, for  $n \in \mathbb{N}_0$  and  $N \in \mathbb{N}$ , we have*

$$(2.7) \quad \begin{aligned} & \frac{4}{\pi^2} \ln(n+1) + c_1 + \sum_{j=1}^{2N} \frac{a_j}{(n+1)^{2j}} \\ & < L_{n/2} < \frac{4}{\pi^2} \ln(n+1) + c_1 + \sum_{j=1}^{2N+1} \frac{a_j}{(n+1)^{2j}}, \end{aligned}$$

where

$$a_j := \frac{8}{\pi^2} \frac{B_{2j}}{2j} (2^{2j-1} - 1) \left( 1 + \sum_{k=1}^j \frac{(-1)^k}{(2k)!} B_{2k} \pi^{2k} \right),$$

and the Bornoulli numbers  $B_k$  is defined by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \quad (|z| < 2\pi).$$

*Proof.* This is Theorem 3.1 of Chen and Choi [8], and also see (3.8) in Chen and Choi [7].  $\square$

In the proof of our inequalities for the constants of Landau and Lebesgue, we also need to use the following simple inequality, which is a consequence of Hermite-Hadamard inequality.

**Lemma 4.** *Let  $f$  be twice derivable, with  $f''$  continuous. If  $f''(x) > 0$ , then*

$$(2.8) \quad \int_a^{a+1} f(x) dx > f(a + 1/2).$$

### 3 The multiple-correction method

First, let us briefly review a so-called *multiple-correction method* presented in our previous paper [6]. Let  $(v(n))_{n \geq 1}$  be a sequence to be approximated. Throughout the paper, we always assume that the following three conditions hold.

**Condition (i).** The initial-correction function  $\eta_0(n)$  satisfies

$$\begin{aligned} \lim_{n \rightarrow \infty} (v(n) - \eta_0(n)) &= 0, \\ \lim_{n \rightarrow \infty} n^{l_0} (v(n) - v(n+1) - \eta_0(n) + \eta_0(n+1)) &= C_0 \neq 0, \end{aligned}$$

for some positive integer  $l \geq 2$ .

**Condition (ii).** The  $k$ -th correction function  $\eta_k(n)$  has the form of  $-\frac{C_{k-1}}{\Phi_k(l_{k-1}; n)}$ , where

$$\lim_{n \rightarrow \infty} n^{l_{k-1}} \left( v(n) - v(n+1) - \sum_{j=0}^{k-1} (\eta_j(n) - \eta_j(n+1)) \right) = C_{k-1} \neq 0,$$

**Condition (iii).** The difference  $(v(1/x) - v(1/x+1) - \eta_0(1/x) + \eta_0(1/x+1))$  is an analytic function in a neighborhood of point  $x = 0$ .

Actually, the *multiple-correction method* is a recursive algorithm. If the assertion

$$\lim_{n \rightarrow \infty} n^{l_{k-1}} \left( v(n) - v(n+1) - \sum_{j=0}^{k-1} (\eta_j(n) - \eta_j(n+1)) \right) = C_{k-1} \neq 0$$

is true, then, it is not difficult to observe

$$\lim_{n \rightarrow \infty} \left( \left( v(n) - v(n+1) - \sum_{j=0}^{k-1} (\eta_j(n) - \eta_j(n+1)) \right) - \frac{C_{k-1}}{n^{l_{k-1}}} \right) = 0.$$

Roughly speaking, the idea of the *multiple-correction method* is to use the polynomial  $\Phi_k(l_{k-1}; n)$  of degree  $l_{k-1}$  instead of  $n^{l_{k-1}}$  for improving the convergence rate. In other words, we view  $n^{l_{k-1}}$  as a special polynomial of degree  $l_{k-1}$  in terms of  $n$ . Here, we note that the polynomial  $\Phi_k(l_{k-1}; n)$  contains  $l_{k-1}$  undetermined parameters  $a_j$  ( $0 \leq j \leq l_{k-1} - 1$ ), and hence, in some cases we hope that we can attain “more gains”.

The initial-correction is a very important step. With this, we hope to further develop the above method starting from the second-correction. To find the proper structure of finite continued fraction, we must try many times by using  $-\frac{C_0}{\Phi_1(l_1; n) + \frac{b_j}{n^j}}$  instead of  $-\frac{C_0}{\Phi_1(l_1; n)}$ , where  $j$  is a positive integer. To do that, we need to begin from  $j = 1$  and try step by step. Once we have found that the convergence rate can be improved for the first positive integer, say  $j_0$ , we use  $\Phi(j_0; n)$  to replace  $n^{j_0}$  immediately, and then, determine all the corresponding coefficients of the polynomial  $\Phi(j_0; n)$ . We continue this process until the desired structure of finite continued fraction is found. It is for this reason that we call it as the *multiple-correction method*.

In addition, to determine all the related coefficients, we often use an appropriate symbolic computation software, which needs a huge of computations. On the other hand, the exact expressions at each occurrence also takes a lot of space. Hence, in this paper we omit some related details for space limitation. For interesting readers, see our previous paper [6].

It is a natural question whether or not *multiple-correction method* can be used to accelerate convergence in some BBP-type or Ramanujan-type series, we hope to return to this topic elsewhere.

## 4 The Landau Constants

**Theorem 1.** *Let sequences  $\text{MC}_k(n)$  be defined as follows:*

$$(4.1) \quad \text{MC}_0(n) := \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) + c_0,$$

$$(4.2) \quad \text{MC}_1(n) := \frac{1}{\pi} \frac{\kappa_1}{(n + \frac{3}{4})^2 + \lambda_1},$$

$$(4.3) \quad \text{MC}_k(n) := \frac{1}{\pi} \frac{\kappa_1}{(n + \frac{3}{4})^2 + \lambda_1} + \sum_{j=2}^k \frac{\kappa_j}{(n + \frac{3}{4})^2 + \lambda_j}, (k \geq 2),$$

where  $c_0$  is determined by (1.4) and

$$\begin{aligned} \kappa_1 &= \frac{11}{192}, & \lambda_1 &= \frac{1541}{7040}, \\ \kappa_2 &= -\frac{89684299}{1040793600}, & \lambda_2 &= \frac{815593360691}{631377464960}, \\ \kappa_3 &= -\frac{791896453750695892475}{691850212268234428416}, & \lambda_3 &= \frac{79124827964452580408836456738931}{23635681749960244849264556808320}. \end{aligned}$$

If we let the  $k$ -th correction error term  $E_k(n)$  be denoted by

$$(4.4) \quad E_k(n) := G(n) - \text{MC}_0(n) - \text{MC}_k(n),$$

then, for all positive integer  $k$ , we have

$$(4.5) \quad \lim_{n \rightarrow \infty} n^{4k+3} (E_k(n) - E_k(n+1)) = (4k+2)C_k,$$

$$(4.6) \quad \lim_{n \rightarrow \infty} n^{4k+2} E_k(n) = C_k,$$

where

$$\begin{aligned} C_1 &= \frac{89684299}{18166579200\pi}, \\ C_2 &= \frac{31675858150027835699}{5605686531912433139712\pi}, \\ C_3 &= \frac{9662255454831353335643376823083291821}{310776980771128296411407710029663436800\pi}. \end{aligned}$$

*Remark 1.* Theorem 1 tells us that it could be possible for us to find a simpler asymptotic expansion than Theorem 2.1 of Chen and Choi [7] for the Landau constants.

*Proof.* Let us consider the initial-correction.

**(Step 1) The initial-correction.** Motivated by inequalities (1.6) and (1.7), we choose  $\text{MC}_0(n) = \frac{1}{\pi} \ln(n + \frac{3}{4}) + c_0$ , and define

$$(4.7) \quad E_0(n) = G(n) - \text{MC}_0(n) = G(n) - \frac{1}{\pi} \ln(n + \frac{3}{4}) - c_0.$$

Then, it follows immediately from (4.7)

$$(4.8) \quad E_0(n) - E_0(n+1) = G(n) - G(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}).$$

Now, by using the duplication formula (Legendre, 1809)

$$(4.9) \quad 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} \Gamma(2z),$$

one can prove

$$(4.10) \quad G(n) - G(n-1) = \frac{(\Gamma(2n+1))^2}{16^2 (\Gamma(n+1))^4} = \left( \frac{(2n)!}{4^n (n!)^2} \right)^2 = \frac{1}{\pi} q(n),$$

where  $q(n)$  is defined by (2.3). Also see Page 739 in Granath [22] or Page 306 in Chen [10]. Combining (4.8) with (4.10) yields

$$(4.11) \quad E_0(n) - E_0(n+1) = -\frac{1}{\pi} q(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}).$$

On one hand, by utilizing Lemma 2 and (2.6), we can obtain that for all positive integer  $j$ ,

$$(4.12) \quad q_2(n) < q_4(n) < \cdots < q_{2j}(n) < q(n) < q_{2j+1}(n) < \cdots < q_3(n) < q_1(n).$$

On the other hand, by using *Mathematica* software, we can attain

$$(4.13) \quad q_9(n) - q_8(n) = O\left(\frac{1}{n^{17}}\right).$$

Now, combining (4.12) and (4.13) gives us

$$(4.14) \quad q(n+1) = q_8(n+1) + O\left(\frac{1}{n^{16}}\right).$$



Again, by making use of the *Mathematica* software, we can expand  $q_8(n+1)$  into a power series in terms of  $n^{-1}$  so that

$$\begin{aligned}
(4.15) \quad q(n+1) &= q_8(n+1) + O\left(\frac{1}{n^{16}}\right) \\
&= \frac{1}{n} - \frac{5}{4} \frac{1}{n^2} + \frac{49}{32} \frac{1}{n^3} - \frac{235}{128} \frac{1}{n^4} + \frac{4411}{2048} \frac{1}{n^5} - \frac{20275}{8192} \frac{1}{n^6} + \frac{183077}{65536} \frac{1}{n^7} \\
&\quad - \frac{815195}{262144} \frac{1}{n^8} + \frac{28754131}{8388608} \frac{1}{n^9} - \frac{125799895}{33554432} \frac{1}{n^{10}} + \frac{1091975567}{268435456} \frac{1}{n^{11}} \\
&\quad - \frac{4702048685}{1073741824} \frac{1}{n^{12}} + \frac{80679143663}{17179869184} \frac{1}{n^{13}} - \frac{346250976095}{68719476736} \frac{1}{n^{14}} \\
&\quad + \frac{2947620308941}{549755813888} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right).
\end{aligned}$$

In addition, it is not difficult to obtain

$$(4.16) \quad -\ln\left(n + \frac{3}{4}\right) + \ln\left(n + \frac{7}{4}\right) = \frac{1}{n} - \frac{5}{4} \frac{1}{n^2} + \frac{79}{48} \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Inserting (4.15) and (4.16) into (4.11) results in

$$(4.17) \quad E_0(n) - E_0(n+1) = \frac{11}{96\pi} \frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Note that the inequalities (1.7) implies  $E_0(\infty) = 0$ . By Lemma 1 again, we obtain

$$(4.18) \quad \lim_{n \rightarrow \infty} n^2 E_0(n) = \frac{11}{192\pi} = C_0 = \kappa_1.$$

**(Step 2) The first-correction.** For simplicity, let

$$(4.19) \quad \text{MC}_1(n) = \frac{1}{\pi} \frac{\kappa_1}{\Phi_1(2; n)} = \frac{1}{\pi} \frac{\kappa_1}{\left(n + \frac{3}{4}\right)^2 + \lambda_1},$$

and define

$$(4.20) \quad E_1(n) := G(n) - \text{MC}_0(n) - \text{MC}_1(n) = E_0(n) - \text{MC}_1(n).$$

Combining (4.11), (4.14) and (4.20), we can obtain

$$\begin{aligned}
(4.21) \quad E_1(n) - E_1(n+1) &= (E_0(n) - E_0(n+1)) - (\text{MC}_1(n) - \text{MC}_1(n+1)) \\
&= -\frac{1}{\pi} q_8(n+1) - \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + \frac{1}{\pi} \ln\left(n + \frac{7}{4}\right) \\
&\quad - \text{MC}_1(n) + \text{MC}_1(n+1) + O\left(\frac{1}{n^{16}}\right).
\end{aligned}$$

By taking advantage of formulae (4.15) and (4.19), and *Mathematica* software, we expand  $E_1(n) - E_1(n+1)$  into power series in terms of  $n^{-1}$ :

$$(4.22) \quad \pi(E_1(n) - E_1(n+1)) = \frac{-\frac{1541}{30720} + \frac{11\lambda_1}{48}}{n^5} + \frac{\frac{7705}{24576} - \frac{275\lambda_1}{192}}{n^6} \\ + \frac{-\frac{183077}{65536} + \frac{275463+973280\lambda_1-59136\lambda_1^2}{172032}}{n^7} + O\left(\frac{1}{n^8}\right).$$

By Lemma 1, the fastest sequence  $(E_1(n))_{n \geq 1}$  is obtained when the first coefficient of this power series vanish. In this case

$$(4.23) \quad \lambda_1 = \frac{1541}{7040},$$

and thus,

$$E_1(n) - E_1(n+1) = \frac{89684299}{3027763200\pi} \frac{1}{n^7} + O\left(\frac{1}{n^8}\right).$$

Now, by Lemma 1 again, we attain

$$(4.24) \quad \lim_{n \rightarrow \infty} n^6 E_1(n) = \frac{89684299}{18166579200\pi} = C_1.$$

**(Step 3) The second-correction.** Let

$$(4.25) \quad \text{MC}_2(n) = \frac{1}{\pi} \frac{\kappa_1}{(n + \frac{3}{4})^2 + \lambda_1} \frac{\kappa_2}{(n + \frac{3}{4})^2 + \lambda_2},$$

and define

$$(4.26) \quad E_2(n) = G(n) - \text{MC}_0(n) - \text{MC}_2(n).$$

Following the way similar to the proof of (4.21), we can obtain

$$(4.27) \quad E_2(n) - E_2(n+1) = -\frac{1}{\pi} q_8(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}) \\ - \text{MC}_2(n) + \text{MC}_2(n+1) + O\left(\frac{1}{n^{16}}\right).$$

By utilizing *Mathematica* software,  $E_2(n) - E_2(n+1)$  can be expanded into power series in terms of  $n^{-1}$

$$(4.28) \quad \pi(E_2(n) - E_2(n+1)) = \frac{\frac{89684299}{3027763200} + \frac{11\kappa_2}{32}}{n^7} - \frac{\frac{89684299}{346030080} + \frac{385\kappa_2}{128}}{n^8} \\ + \frac{\frac{2961426180353}{2283798528000} + \frac{120119\kappa_2}{7680} - \frac{11\kappa_2\lambda_2}{24}}{n^9} \\ + \frac{-\frac{988371602353}{203004313600} - \frac{129357\kappa_2}{2048} + \frac{165\kappa_2\lambda_2}{32}}{n^{10}} + O\left(\frac{1}{n^{12}}\right) \\ + \frac{\frac{65353200785578639}{4287451103232000} + \frac{6308113241\kappa_2}{28835840} - \frac{207019\kappa_2\lambda_2}{6144} + \frac{55\kappa_2\lambda_2^2}{96} - \frac{55\kappa_2^2}{96}}{n^{11}}.$$

The fastest sequence  $(E_2(n))_{n \geq 1}$  is obtained by enforcing the first four coefficients of this power series to be zeros. In this case

$$(4.29) \quad \kappa_2 = -\frac{89684299}{1040793600} \quad \text{and} \quad \lambda_2 = \frac{815593360691}{631377464960},$$

and hence,

$$E_2(n) - E_2(n+1) = \frac{158379290750139178495}{2802843265956216569856\pi} \frac{1}{n^{11}} + O\left(\frac{1}{n^{12}}\right).$$

Combining this with Lemma 1 leads to

$$(4.30) \quad \lim_{n \rightarrow \infty} n^{10} E_2(n) = \frac{31675858150027835699}{5605686531912433139712\pi} := C_2.$$

**(Step 4) The third-correction.** Let

$$(4.31) \quad \text{MC}_3(n) := \frac{1}{\pi} \frac{\kappa_1}{(n + \frac{3}{4})^2 + \lambda_1 + (n + \frac{3}{4})^2 + \lambda_2 + (n + \frac{3}{4})^2 + \lambda_3},$$

If we define

$$(4.32) \quad E_3(n) = G(n) - \text{MC}_0(n) - \text{MC}_3(n).$$

then, by using the same approach as Step 3, we can prove

$$(4.33) \quad E_3(n) - E_3(n+1) = -\frac{1}{\pi} q_8(n+1) - \frac{1}{\pi} \ln(n + \frac{3}{4}) + \frac{1}{\pi} \ln(n + \frac{7}{4}) \\ - \text{MC}_3(n) + \text{MC}_3(n+1) + O\left(\frac{1}{n^{16}}\right),$$

and thus, find

$$\kappa_3 = -\frac{791896453750695892475}{691850212268234428416}, \lambda_3 = \frac{79124827964452580408836456738931}{23635681749960244849264556808320}.$$

Similarly, using the *Mathematica* software can produce

$$E_3(n) - E_3(n+1) = \frac{67635788183819473349503637761583042747}{155388490385564148205703855014831718400} \frac{1}{n^{15}} + O\left(\frac{1}{n^{16}}\right).$$

Finally, by Lemma 1 we have

$$(4.34) \quad \lim_{n \rightarrow \infty} n^{14} E_3(n) = \frac{9662255454831353335643376823083291821}{310776980771128296411407710029663436800\pi} := C_3.$$

This completes the proof of Theorem 1. □

**Theorem 2.** Let  $\text{MC}_2(n)$  be defined in Theorem 1. Then, for all integer  $n \geq 0$ , we have

$$(4.35) \quad \frac{C_2}{(n + \frac{7}{4})^{10}} < G(n) - \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) - c_0 - \text{MC}_2(n) < \frac{C_2}{(n + \frac{3}{4})^{10}},$$

where  $C_2 = \frac{31675858150027835699}{5605686531912433139712\pi}$ .

*Remark 2.* In fact, Theorem 2 implies that  $E_2(n)$  is a strictly decreasing function of  $n$ . In addition, it should be possible to establish many these types of inequalities by using the same method of Theorem 2.

*Proof.* First, it is not difficult to verify that (4.35) is true for  $n = 0$ . Hence, in the following we only need to prove that (4.35) holds for  $n \geq 1$ . For notational simplicity, we let  $D_2 = \frac{158379290750139178495}{254803933268746960896}$ , and

$$(4.36) \quad E_2(n) = G(n) - \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) - c_0 - \text{MC}_2(n).$$

Then, it follows from (4.10)

$$(4.37) \quad \begin{aligned} E_2(n) - E_2(n+1) &= -\frac{1}{\pi} q(n+1) - \frac{1}{\pi} \ln \left( n + \frac{3}{4} \right) - \text{MC}_2(n) \\ &\quad + \frac{1}{\pi} \ln \left( n + \frac{7}{4} \right) + \text{MC}_2(n+1). \end{aligned}$$

If we let

$$(4.38) \quad U(x) = -\frac{1}{\pi} q_8(x+1) - \frac{1}{\pi} \ln \left( x + \frac{3}{4} \right) - \text{MC}_2(x) + \frac{1}{\pi} \ln \left( x + \frac{7}{4} \right) + \text{MC}_2(x+1),$$

$$(4.39) \quad V(x) = -\frac{1}{\pi} q_7(x+1) - \frac{1}{\pi} \ln \left( x + \frac{3}{4} \right) - \text{MC}_2(x) + \frac{1}{\pi} \ln \left( x + \frac{7}{4} \right) + \text{MC}_2(x+1),$$

then, combining (4.12) and (4.37)-(4.39) yields

$$(4.40) \quad V(n) < E_2(n) - E_2(n+1) < U(n).$$

In the following, we establish the lower bound of  $V(n)$  and the upper bound of  $U(n)$ . First, by using the *Mathematica* software, we can obtain

$$(4.41) \quad -U'(x) - \frac{D_2}{\pi(x + \frac{5}{4})^{12}} = -\frac{1}{\pi} \frac{\Psi_1(32; n)}{75937489649280(3+4n)(5+4n)^{12}(7+4n)\Psi_2(32; n)} < 0.$$

Noticing  $U(+\infty) = 0$ , and utilizing (4.41) and Lemma 4, we have

$$(4.42) \quad \begin{aligned} U(n) &= \int_n^\infty -U'(x) dx < \int_n^\infty \frac{D_2}{\pi(x + \frac{5}{4})^{12}} dx = \frac{D_2}{11\pi} \frac{1}{(n + \frac{5}{4})^{11}} \\ &< \frac{D_2}{11\pi} \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{1}{x^{11}} dx. \end{aligned}$$

Similarly, we can attain

$$(4.43) \quad -V'(x) - \frac{D_2}{\pi(x + \frac{7}{4})^{12}} = \frac{1}{\pi} \frac{\Psi_3(30; n)}{75937489649280(3 + 4n)(5 + 4n)^2(7 + 4n)^{12}\Psi_4(28; n)} > 0.$$

Therefore, integrating (4.43) with  $V(+\infty) = 0$  results in

$$(4.44) \quad \begin{aligned} V(n) &= \int_n^\infty -V'(x)dx > \int_n^\infty \frac{D_2}{\pi(x + \frac{7}{4})^{12}}dx = \frac{D_2}{11\pi} \frac{1}{(n + \frac{7}{4})^{11}} \\ &> \frac{D_2}{11\pi} \int_{n+\frac{7}{4}}^{n+\frac{11}{4}} \frac{1}{x^{11}}dx, \end{aligned}$$

which, along with  $E_2(\infty) = 0$  and (4.40) gives us

$$(4.45) \quad \begin{aligned} E_2(n) &= \sum_{m=n}^\infty (E_2(m) - E_2(m+1)) > \sum_{m=n}^\infty \frac{D_2}{11\pi} \int_{m+\frac{7}{4}}^{m+\frac{11}{4}} \frac{1}{x^{11}}dx \\ &= \frac{D_2}{11\pi} \int_{n+\frac{7}{4}}^\infty \frac{1}{x^{11}}dx = \frac{D_2}{110\pi} \frac{1}{(n + \frac{7}{4})^{10}}. \end{aligned}$$

Similarly, combining (4.42) with (4.40) yields

$$(4.46) \quad \begin{aligned} E_2(n) &= \sum_{m=n}^\infty (E_2(m) - E_2(m+1)) < \sum_{m=n}^\infty \frac{D_2}{11\pi} \int_{m+\frac{3}{4}}^{m+\frac{7}{4}} \frac{1}{x^{11}}dx \\ &= \frac{D_2}{11\pi} \int_{n+\frac{3}{4}}^\infty \frac{1}{x^{11}}dx = \frac{D_2}{110\pi} \frac{1}{(n + \frac{3}{4})^{10}}. \end{aligned}$$

This finishes the proof of Theorem 2. □

## 5 The Lebesgue constants

For the Lebesgue constants, we will prove the following hybrid-type finite continued fraction approximations, which has a structure similar to that of the Landau constants.

**Theorem 3.** *Let the initial-correction function be given by  $\text{MC}_0(n) = \frac{4}{\pi^2} \ln(n+1) + c_1$ , where  $c_1$  is defined by (1.15). If we let the  $k$ -th correction function  $\text{MC}_k(n)$  for  $k \geq 1$  be defined by*

$$(5.1) \quad \text{MC}_1(n) := \frac{\rho_1}{(n+1)^2 + \varrho_1},$$

$$(5.2) \quad \text{MC}_k(n) := \frac{\rho_1}{(n+1)^2 + \varrho_1} + \mathbf{K}_{j=2}^k \frac{\rho_j}{(n+1)^2 + \varrho_j}, (k \geq 2).$$

where

$$\begin{aligned}\rho_1 &= \frac{12 - \pi^2}{18\pi^2}, \\ \varrho_1 &= \frac{7(-720 + 60\pi^2 + \pi^4)}{600(-12 + \pi^2)}, \\ \rho_2 &= -\frac{7515244800 - 1252540800\pi^2 + 46937520\pi^4 + 65640\pi^6 + 23797\pi^8}{52920000(-12 + \pi^2)^2}, \\ \varrho_2 &= 7(-36262162944000 + 9065540736000\pi^2 - 720128102400\pi^4 + 16206350400\pi^6 \\ &\quad + 117169920\pi^8 + 288540\pi^{10} + 230953\pi^{12}) / (600(-90182937600 + 22545734400\pi^2 \\ &\quad - 1815791040\pi^4 + 46149840\pi^6 - 219924\pi^8 + 23797\pi^{10})).\end{aligned}$$

and the corresponding  $k$ -th correction error term  $E_k(n)$  be defined by

$$(5.3) \quad E_k(n) := L_{n/2} - \text{MC}_0(n) - \text{MC}_k(n),$$

then, for all positive integer  $k$ , we have

$$(5.4) \quad \lim_{n \rightarrow \infty} n^{4k+3} (E_k(n) - E_k(n+1)) = (4k+2)C_k,$$

$$(5.5) \quad \lim_{n \rightarrow \infty} n^{4k+2} E_k(n) = C_k,$$

where

$$\begin{aligned}C_1 &= \frac{-7515244800 + 1252540800\pi^2 - 46937520\pi^4 - 65640\pi^6 - 23797\pi^8}{952560000\pi^2(-12 + \pi^2)}, \\ C_2 &= (7633889107527073628160000 - 1908472276881768407040000\pi^2 + 146687085183488661504000\pi^4 \\ &\quad - 3184401328004768256000\pi^6 + 50811629937851059200\pi^8 - 5860796365392595200\pi^{10} \\ &\quad + 73433337261096960\pi^{12} - 2698623258901920\pi^{14} - 13989723377364\pi^{16} - 552278517605\pi^{18}) \\ &\quad / (97592743987200\pi^2(7515244800 - 1252540800\pi^2 + 46937520\pi^4 + 65640\pi^6 + 23797\pi^8)).\end{aligned}$$

*Proof.* Since the proof of Theorem 3 is very similar to that of Theorem 1, we only outline the idea of the proof here. First, we recall that

$$(5.6) \quad E_k(n) := L_{n/2} - \text{MC}_0(n) - \text{MC}_k(n).$$

For every positive integer  $M$ , we let

$$(5.7) \quad W_M(n) := \sum_{j=1}^M \frac{a_j}{(n+1)^{2j}},$$

where  $a_j$  is given in Lemma 3. It is not hard to see that  $a_j > 0$  for odd  $j = 1, 3, \dots$ , and  $a_j < 0$  for even  $j = 2, 4, \dots$ . It follows easily from Lemma 3 and (5.6) that

$$(5.8) \quad E_k(n) = W_{2k+1}(n) - \text{MC}_k(n) + O(n^{4k+4}),$$

$$(5.9) \quad E_k(n) - E_k(n+1) = W_{2k+1}(n) - \text{MC}_k(n) - W_{2k+1}(n+1) + \text{MC}_k(n+1) + O(n^{4k+4}).$$

Hence, it suffices for us to approximate  $W_{2k+1}(n)$ . Similar to the proof of Theorem 1, we expand  $W_{2k+1}(n) - \text{MC}_k(n) - W_{2k+1}(n+1) + \text{MC}_k(n+1)$  into a power series in terms of  $n^{-1}$ , and then check (5.5) holds.  $\square$

The main purpose of this section is to prove the following theorem, which corresponds to inequalities (1.10) in the case of Landau constants.

**Theorem 4.** *Let  $\text{MC}_1(n)$  be defined in Theorem 3. Then, for all integer  $n \geq 0$ , we have*

$$(5.10) \quad \frac{C_1}{(n + \frac{13}{8})^6} < L_{n/2} - \frac{4}{\pi^2} \ln(n+1) - c_1 - \text{MC}_1(n) < \frac{C_1}{(n + \frac{5}{8})^6},$$

where  $C_1 = \frac{-7515244800 + 1252540800\pi^2 - 46937520\pi^4 - 65640\pi^6 - 23797\pi^8}{952560000\pi^2(-12 + \pi^2)} > 0$ .

*Remark 3.* In fact, Theorem 4 implies that  $E_1(n)$  is a strictly decreasing function of  $n$ .

*Proof.* First, since  $L_{n/2} = 1$  for  $n = 0$ , one may verify that (5.10) is true for  $n = 0$ . When  $n = 1$ , it follows from Lemma 3 and (5.7) that

$$\text{MC}_0(n) + W_4(n) < L_{n/2} < \text{MC}_0(n) + W_3(n),$$

it is not difficult to verify that (5.10) is also true for  $n = 1$ . Hence, in the following we only need to prove that (5.10) holds for  $n \geq 2$ . By (5.3) and Lemma 3 we have

$$(5.11) \quad \begin{aligned} E_1(n) - E_1(n+1) &= (L_{n/2} - \text{MC}_0(n)) - \text{MC}_1(n) - (L_{(n+1)/2} - \text{MC}_0(n+1)) + \text{MC}_1(n+1) \\ &< W_3(n) - \text{MC}_1(n) - W_4(n+1) + \text{MC}_1(n+1) \\ &= (W_3(n) - \text{MC}_1(n) - W_3(n+1) + \text{MC}_1(n+1)) - \frac{a_4}{(n+2)^8}. \end{aligned}$$

Similarly, we also have

$$(5.12) \quad E_1(n) - E_1(n+1) > (W_3(n) - \text{MC}_1(n) - W_3(n+1) + \text{MC}_1(n+1)) + \frac{a_4}{(n+1)^8}.$$

For notational simplicity, we let  $D_1 = 42C_1$ . Now we define for  $x \geq 1$

$$(5.13) \quad F(x) := W_3(x) - \text{MC}_1(x) - W_3(x+1) + \text{MC}_1(x+1),$$

$$(5.14) \quad U(x) := \frac{D_1}{(x + \frac{5}{4})^8}, \quad V(x) := \frac{D_1}{(x + \frac{3}{2})^8}.$$

In the following, we establish the upper bound and lower bounds of  $F(x)$ , respectively. By using *Mathematica* software, one can check

$$(5.15) \quad -F'(x) - U(x) = \frac{P_1(21; x)}{19845000\pi^2(-12 + \pi^2)(1+x)^7(2+x)^7(5+4x)^8\Psi_1(4; x)\Psi_2(4; x)},$$

where polynomial  $P_1(21; x)$  may be expressed as

$$(5.16) \quad P_1(21; x) = (x-1) \left( \frac{b_{-1}}{x-1} + b_0 + b_1x + \cdots + b_{20}x^{20} \right).$$

By using *Mathematica* software again, it is not hard to verify that all coefficients  $b_j$  ( $-1 \leq j \leq 20$ ) are positive. Thus, the inequality  $P_1(21; x) > 0$  holds for  $x \geq 1$ . Noticing that  $-12 + \pi^2 < 0$ , one obtains by Lemma 4

$$(5.17) \quad -F'(x) < U(x), \quad x \geq 1,$$

$$(5.18) \quad F(n) = \int_n^{+\infty} -F'(x)dx \leq \int_n^{+\infty} U(x)dx = \frac{D_1}{7} \frac{1}{(n + \frac{5}{4})^7} \leq \frac{D_1}{7} \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{dx}{x^7}.$$

On the other hand, we can prove by using *Mathematica* software

$$(5.19) \quad -F'(x) - V(x) = \frac{P_2(20; x)}{19845000\pi^2(-12 + \pi^2)(1+x)^7(2+x)^7(3+2x)^8\Psi_1(4; x)\Psi_2(4; x)},$$

where

$$(5.20) \quad P_2(20; x) = d_0 + d_1x + \cdots + d_{20}x^{20},$$

and all coefficients  $d_j$  ( $0 \leq j \leq 20$ ) are negative. Thus, this yields

$$(5.21) \quad -F'(x) > V(x), \quad x \geq 1,$$

$$(5.22) \quad F(n) = \int_n^{+\infty} -F'(x)dx \geq \int_n^{+\infty} V(x)dx = \frac{D_1}{7} \frac{1}{(n + \frac{3}{2})^7} \geq \frac{D_1}{7} \int_{n+\frac{3}{2}}^{n+\frac{5}{2}} \frac{dx}{x^7}.$$

Combining (5.11), (5.12), (5.13), (5.18) and (5.22) gives

$$(5.23) \quad \frac{D_1}{7} \int_{n+\frac{3}{2}}^{n+\frac{5}{2}} \frac{dx}{x^7} + \frac{a_4}{(n+1)^8} < E_1(n) - E_1(n+1) < \frac{D_1}{7} \int_{n+\frac{3}{4}}^{n+\frac{7}{4}} \frac{dx}{x^7} - \frac{a_4}{(n+2)^8}.$$

By adding the estimates from  $n$  to  $\infty$  and noticing  $E_1(\infty) = 0$ , we attain

$$(5.24) \quad \frac{D_1}{7} \int_{n+\frac{3}{2}}^{\infty} \frac{dx}{x^7} + a_4 \sum_{m=n}^{\infty} \frac{1}{(m+1)^8} < E_1(n) < \frac{D_1}{7} \int_{n+\frac{3}{4}}^{\infty} \frac{dx}{x^7} - a_4 \sum_{m=n}^{\infty} \frac{1}{(m+2)^8}.$$

By Lemma 4 again, one has

$$(5.25) \quad \sum_{m=n}^{\infty} \frac{1}{(m+2)^8} \leq \sum_{m=n}^{\infty} \int_{m+\frac{3}{2}}^{m+\frac{5}{2}} \frac{dx}{x^8} = \frac{1}{7(n + \frac{3}{2})^7}.$$

On the other hand, one has the following trivial estimate

$$(5.26) \quad \sum_{m=n}^{\infty} \frac{1}{(m+1)^8} \geq \int_{n+1}^{\infty} \frac{dx}{x^8} = \frac{1}{7(n+1)^7}.$$



Substituting the above two estimates into (5.24) produces

$$(5.27) \quad \frac{C_1}{(n + \frac{3}{2})^6} + \frac{a_4}{7(n+1)^7} < E_1(n) < \frac{C_1}{(n + \frac{3}{4})^6} - \frac{a_4}{7(n + \frac{3}{2})^7}.$$

By using *Mathematica* software, it is not difficult to check

$$(5.28) \quad \frac{C_1}{(n + \frac{5}{8})^6} - \left( \frac{C_1}{(n + \frac{3}{4})^6} - \frac{a_4}{7(n + \frac{3}{2})^7} \right) \\ = \frac{P_3(12; n)}{29767500\pi^2(-12 + \pi^2)(3 + 2n)^7(3 + 4n)^6(5 + 8n)^6},$$

where

$$(5.29) \quad P_3(12; n) = (n - 1) \left( \frac{\theta_{-1}}{n - 1} + \theta_0 + \theta_1 n + \cdots + \theta_{11} n^{11} \right).$$

By utilizing *Mathematica* software again, we observe that all coefficients  $\theta_j$  ( $-1 \leq j \leq 11$ ) are negative. Hence

$$(5.30) \quad \frac{C_1}{(n + \frac{5}{8})^6} - \left( \frac{C_1}{(n + \frac{3}{4})^6} - \frac{a_4}{7(n + \frac{3}{2})^7} \right) \geq 0, \quad (n \geq 1).$$

Similarly, one may check

$$(5.31) \quad \left( \frac{C_1}{(n + \frac{3}{2})^6} + \frac{a_4}{7(n+1)^7} \right) - \frac{C_1}{(n + \frac{13}{8})^6} \\ = \frac{(n - 2) \left( \frac{\vartheta_{-1}}{n - 2} + \vartheta_0 + \vartheta_1 n + \cdots + \vartheta_{11} n^{11} \right)}{3810240000\pi^2(-12 + \pi^2)(1 + n)^7(3 + 2n)^6(13 + 8n)^6},$$

and all coefficients  $\vartheta_j$  ( $-1 \leq j \leq 11$ ) are negative. Thus, we obtain

$$(5.32) \quad \left( \frac{C_1}{(n + \frac{3}{2})^6} + \frac{a_4}{7(n+1)^7} \right) - \frac{C_1}{(n + \frac{13}{8})^6} > 0, \quad (n \geq 2).$$

Finally, Theorem 4 follows from (5.27), (5.30) and (5.32) immediately.  $\square$

## 6 The Euler-Mascheroni constant

The Euler constant was first introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$(6.1) \quad \gamma(n) := \sum_{m=1}^n \frac{1}{m} - \ln n.$$

It is also known as the Euler-Mascheroni constant. There are many famous unsolved problems about the nature of this constant. See e.g. the survey papers or books of R.P. Brent and P. Zimmermann [4], Dence and Dence [14], Havil [25] and Lagarias [26]. For example, a long-standing open problem is whether or not it is a rational number.

In fact, the sequence  $(\gamma(n))_{n \in \mathbb{N}}$  converges very slowly toward  $\gamma$ , like  $(2n)^{-1}$ . Up to now, many authors are preoccupied to improve its rate of convergence. See e.g. [11, 14, 15, 20, 21, 29, 30, 31, 35] and references therein. Let  $R_1(n) = \frac{a_1}{n}$  and for  $k \geq 2$

$$(6.2) \quad R_k(n) := \frac{a_1}{n + \frac{a_2 n}{n + \frac{a_3 n}{n + \frac{a_4 n}{\ddots \frac{a_k n}{n + a_k}}}}},$$

where  $(a_1, a_2, a_4, a_6, a_8, a_{10}, a_{12}) = (\frac{1}{2}, \frac{1}{6}, \frac{3}{5}, \frac{79}{126}, \frac{7230}{6241}, \frac{4146631}{3833346}, \frac{306232774533}{179081182865})$ ,  $a_{2k+1} = -a_{2k}$  for  $1 \leq k \leq 6$ , and

$$(6.3) \quad r_k(n) := \sum_{m=1}^n \frac{1}{m} - \ln n - R_k(n).$$

Lu [29] introduced the continued fraction method to investigate this problem, and showed

$$(6.4) \quad \frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}.$$

Xu and You [44] continued Lu's work to find  $a_5, \dots, a_{13}$  with the help of *Mathematica* software, and obtained

$$(6.5) \quad \lim_{n \rightarrow \infty} n^{k+1} (r_k(n) - \gamma) = C'_k,$$

where  $(C'_1, \dots, C'_{13}) = (-\frac{1}{12}, -\frac{1}{72}, \frac{1}{120}, \frac{1}{200}, -\frac{79}{25200}, -\frac{6241}{3175200}, \frac{241}{105840}, \frac{58081}{22018248}, -\frac{262445}{91974960}, -\frac{2755095121}{892586949408}, \frac{20169451}{3821257440}, \frac{406806753641401}{45071152103463200}, -\frac{71521421431}{5152068292800})$ . Hence the rate of the convergence of the sequence  $(r_k(n))_{n \in \mathbb{N}}$  is  $n^{-(k+1)}$ . Moreover, they improved (6.4) to

$$(6.6) \quad C'_{10} \frac{1}{(n+1)^{11}} < \gamma - r_{10}(n) < C'_{10} \frac{1}{n^{11}},$$

$$(6.7) \quad C'_{11} \frac{1}{(n+1)^{12}} < r_{11}(n) - \gamma < C'_{11} \frac{1}{n^{12}}.$$

The purpose of this section is to further refine the works of Lu [29] and Xu and You [44] by using the *multiple-correction method*, and prove the following theorem.

**Theorem 5.** *For every positive integer  $k$ , let the  $k$ -th correction function  $\text{MC}_k(n)$  be defined by*

$$(6.8) \quad \text{MC}_1(n) := \frac{a_1}{n + b_1},$$

$$(6.9) \quad \text{MC}_k(n) := \frac{a_1}{n + b_1 + \prod_{j=2}^k \frac{a_j}{n + b_j}}, (k \geq 2),$$

where

$$\begin{aligned}
a_1 &= \frac{1}{2}, & b_1 &= \frac{1}{6}, \\
a_2 &= \frac{1}{36}, & b_2 &= \frac{13}{30}, \\
a_3 &= \frac{9}{25}, & b_3 &= \frac{17}{630}, \\
a_4 &= \frac{6241}{15876}, & b_4 &= \frac{417941}{786366}, \\
a_5 &= \frac{52272900}{38950081}, & b_5 &= -\frac{1835967509}{23923912386}, \\
a_6 &= \frac{17194548650161}{14694541555716}, & b_6 &= \frac{431312596940299603}{686480136010816290}, \\
a_7 &= \frac{93778512198179213368089}{32070070056327569608225}, & b_7 &= -\frac{75178865368857369613934863}{437108607837436422694763190}, \\
a_8 &= \frac{14093175882028689333655328914081}{5957702453097198927838844740836}, & b_8 &= \frac{152838545298199920648591716358691154137}{212256305311307139071033336233757302422}.
\end{aligned}$$

If we let the  $k$ -th correction error term  $E_k(n)$  be defined by

$$(6.10) \quad E_k(n) := \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma - \text{MC}_k(n),$$

then, for all positive integer  $k$ , we have

$$(6.11) \quad \lim_{n \rightarrow \infty} n^{2k+2} (E_k(n) - E_k(n+1)) = (2k+1)C_k,$$

$$(6.12) \quad \lim_{n \rightarrow \infty} n^{2k+1} E_k(n) = C_k,$$

where  $(C_1, C_2, C_3, \dots, C_6) = (-\frac{1}{72}, \frac{1}{200}, -\frac{6241}{3175200}, \frac{58081}{22018248}, -\frac{2755095121}{892586949408}, \frac{406806753641401}{45071152103463200})$ ,  $C_7 = -\frac{5115313723510706087761}{239581189590134660611200}$  and  $C_8 = \frac{26329150006913625404731665769}{241842252367746831359300280968}$ .

*Remark 4.* For comparison with (6.2), Theorem 5 is more convenient for us to find  $a_k$  and  $b_k$ , since the parameter  $a_k$  and the variable  $n$  in Lu's continued fraction need to be iterated more times than ours in the recursive algorithm.

*Remark 5.* It is interesting to note that we have  $|C_k| < 1$  for all positive integers  $k$  ( $1 \leq k \leq 8$ ), and that the correction function  $\text{MC}_k(n)$  for all positive integers  $k$  is a rational function in the form of  $\frac{P_{k-1}(n)}{Q_k(n)}$  with  $P_{k-1}(x), Q_k(x) \in \mathbb{Q}[x]$ . Therefore, the rate of convergence for the  $(E_k(n))_{k \geq 1}$  is much faster than the geometric series if we can repeat the multiple-correction infinite times. Hence, Theorem 5 predicts that it may be possible for us to find a more rapidly or BBP-type series expansion for the Euler-Mascheroni constant in the future.

*Proof of Theorem 5:* This proof consists of the following steps.

**(Step 1) The initial-correction.** We choose  $\text{MC}_0(n) = 0$ , in other words, we don't need the initial-correction, and let

$$(6.13) \quad E_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma - \text{MC}_0(n) = \sum_{m=1}^n \frac{1}{m} - \ln n.$$

It is not difficult to check that

$$(6.14) \quad \lim_{n \rightarrow \infty} n^2 (E_0(n) - E_0(n+1)) = \lim_{n \rightarrow \infty} n^2 \left( \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} \right) = \frac{1}{2}.$$

Using Lemma 1 and Noting that  $E_0(\infty) = 0$ , we have

$$(6.15) \quad \lim_{n \rightarrow \infty} n E_0(n) = \frac{1}{2} =: a_1 = C_0.$$

**(Step 2) The first-correction.** We let

$$(6.16) \quad \text{MC}_1(n) = \frac{a_1}{\Phi_1(1; n)} = \frac{a_1}{n + b_1},$$

and define

$$(6.17) \quad E_1(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma - \text{MC}_1(n).$$

By making use of *Mathematica* software, we expand the difference  $E_1(n) - E_1(n+1)$  into a power series in terms of  $n^{-1}$ :

$$(6.18) \quad \begin{aligned} E_1(n) - E_1(n+1) &= \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} - \text{MC}_1(n) + \text{MC}_1(n+1) \\ &= \frac{-\frac{1}{6} + b_1}{n^3} + \frac{1 - 6b_1 - 6b_1^2}{4n^4} + O\left(\frac{1}{n^5}\right). \end{aligned}$$

By Lemma 1, the fastest sequence  $E_1(n)_{n \geq 1}$  is obtained by enforcing the first coefficient of this power series to be zero. In this case,  $b_1 = \frac{1}{6}$  and thus,

$$(6.19) \quad \lim_{n \rightarrow \infty} n^4 (E_1(n) - E_1(n+1)) = -\frac{1}{24}.$$

Applying Lemma 1 again yields

$$(6.20) \quad \lim_{n \rightarrow \infty} n^3 E_1(n) = -\frac{1}{72} := C_1.$$

**(Step 3) The second-correction.** We choose

$$(6.21) \quad \text{MC}_2(n) = \frac{a_1}{n + b_1} + \frac{a_2}{n + b_2}$$

and define

$$(6.22) \quad E_2(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma - \text{MC}_2(n).$$

Similar to the first-correction, *Mathematica* software helps us find the power series of  $E_2(n) - E_2(n+1)$  in terms of  $n^{-1}$ :

$$(6.23) \quad \begin{aligned} E_2(n) - E_2(n+1) &= \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} - \text{MC}_2(n) + \text{MC}_2(n+1) \\ &= \frac{-\frac{1}{24} + \frac{3a_2}{2}}{n^4} + \frac{\frac{17}{135} - \frac{11a_2}{3} - 2a_2b_2}{n^5} \\ &\quad + \frac{-641 + 17820a_2 - 6480a_2^2 + 15120a_2b_2 + 6480a_2b_2^2}{2592n^6} + O\left(\frac{1}{n^7}\right). \end{aligned}$$

In order to obtain the fastest convergence of the sequence from (6.23), we enforce

$$\begin{cases} -\frac{1}{24} + \frac{3a_2}{2} = 0, \\ \frac{17}{135} - \frac{11a_2}{3} - 2a_2b_2 = 0, \end{cases}$$

which is equivalent to

$$(6.24) \quad a_2 = \frac{1}{36} \quad \text{and} \quad b_2 = \frac{13}{30}.$$

Therefore, we attain

$$(6.25) \quad E_2(n) - E_2(n+1) = \frac{1}{40} \frac{1}{n^6} + O\left(\frac{1}{n^7}\right).$$

Now by Lemma 1 again, we have

$$(6.26) \quad \lim_{n \rightarrow \infty} n^6 E_2(n) = \frac{1}{200} := C_2.$$

Since the derivations from the third-correction to the eighth-correction are very similar, here we only give the proof of the eighth-correction.

**(Step 9) The eighth-correction.** We let

$$(6.27) \quad \text{MC}_8(n) = \frac{a_1}{n + b_1} + \sum_{j=2}^8 \frac{a_j}{n + b_j},$$

and define

$$(6.28) \quad E_8(n) = \sum_{m=1}^n \frac{1}{m} - \ln n - \gamma - \text{MC}_8(n).$$

Again, we resort to *Mathematica* software to expand the difference  $E_8(n) - E_8(n+1)$  into a power series in terms of  $n^{-1}$ :

$$\begin{aligned}
(6.29) \quad & E_8(n) - E_8(n+1) \\
&= \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} - \text{MC}_8(n) + \text{MC}_8(n+1) \\
&= -\frac{5115313723510706087761}{15972079306008977374080} + \frac{1220420260924203a_8}{9014230420692640} + O\left(\frac{1}{n^{19}}\right) \\
&\quad + \frac{\phi_1}{427954568312084496528235420668724754292663281100n^{17}} \\
&\quad + \frac{\phi_2}{3932494501068339337733207321439015145340880276359961361938547200n^{18}},
\end{aligned}$$

where

$$\begin{aligned}
\phi_1 &= 1651455193723916359597152051576300956283985319207 \\
&\quad - 653628703213874127417970758864749834347105473339b_8 \\
&\quad - 61802656649700006308879054758037254288686256518a_8b_8 \\
\phi_2 &= -84020030461785776411416922144974524623382664719083303798933256161 \\
&\quad + 34897177003226720866547593123620552491030169390762369825972005484a_8 \\
&\quad + 6683314557607540897917106944093661063444345868437190779381411152a_8b_8 \\
&\quad + 603401714833886490440852089976344096105815206107927672101641232b_8^2a_8 \\
&\quad - 603401714833886490440852089976344096105815206107927672101641232a_8^2.
\end{aligned}$$

By enforcing  $a$  and  $b$  in (6.29) to satisfy the following condition:

$$\begin{cases} -\frac{5115313723510706087761}{15972079306008977374080} + \frac{1220420260924203a_8}{9014230420692640} = 0, \\ \phi_1 = 0, \end{cases}$$

i.e.,

$$a_8 = \frac{14093175882028689333655328914081}{5957702453097198927838844740836}, b_8 = \frac{152838545298199920648591716358691154137}{212256305311307139071033336233757302422},$$

we obtain

$$(6.30) \quad E_8(n) - E_8(n+1) = \frac{26329150006913625404731665769}{14226014845161578315252957704} \frac{1}{n^{18}} + O\left(\frac{1}{n^{19}}\right).$$

Now by Lemma 1 again, we finally attain

$$(6.31) \quad \lim_{n \rightarrow \infty} n^{17} E_2(n) = \frac{26329150006913625404731665769}{241842252367746831359300280968} := C_8.$$

This completes the proof of Theorem 5. □

**Appendix.** For the reader's convenience, here we give some more constants in our theorems. In fact, one can use *Mathematica* command "Together" and "Coefficient" to find more constants.

$$\begin{aligned}
\kappa_4 &= -\frac{382149699786434954423663192287642772100258949239}{69454986539981103777883874787703791756725862400}, \\
\kappa_5 &= -\frac{67932766531075103743956191388015599579295382477479617083158574631741016483865590781675}{4001606232950669353994900572807255080083296908405703258931156260080111590006888051712}, \\
\lambda_4 &= \frac{3047183642643398321446537081211433153790774725879204120678621187}{476180753216552458418280167270798333222960626510492964456863360}, \\
\rho_3 &= -25(91606669290324883537920000 - 30535556430108294512640000\pi^2 \\
&\quad + 3668717299083632345088000\pi^4 - 184899901119545880576000\pi^6 \\
&\quad + 3794140887258980966400\pi^8 - 121141186322562201600\pi^{10} \\
&\quad + 6741996412525758720\pi^{12} - 105816816367920000\pi^{14} \\
&\quad + 2530746578373552\pi^{16} + 7362381166104\pi^{18} + 552278517605\pi^{20})/ \\
&\quad (2561328(7515244800 - 1252540800\pi^2 + 46937520\pi^4 + 65640\pi^6 + 23797\pi^8)^2) \\
a_9 &= \frac{38559153745620009525389781729558359566448528400}{7562099567591782725341311886983340261624011969}, \\
b_9 &= -\frac{4311810252990337765692084981855831368824641381949822699}{16443847302827668255907904514549005300064801885045499646}, \\
a_{10} &= \frac{142440816556951082015748637112875838629364253067475021984438416009}{35757280329598209749962500807452853821298673049007961786630549764}, \\
b_{10} &= \frac{106368952896545249534816650756049857087954719240036868272942545496721248718541}{131638955807463173095557478201986471603558078059274032167358944757539476827550}.
\end{aligned}$$

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